The Theory of Designed Experiments

10. Multi-Stratum Designs

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Multi-Stratum Designs

There are many situations in which different factors must be applied to units of different sizes.

Example: In an experiment on pastry dough, 3 baking temperatures and 2 recipes have to be compared. The temperature can only be varied between runs of the oven and there are sufficient resources to make 12 such runs. However the two recipes can both be used in the same run of the oven. There are two possible solutions.

1. use runs of the oven as units;
2. use runs as "units" for comparing temperatures and half-runs as units for comparing recipes.

Multi-Stratum Designs

The first design gives the following analysis of variance:

Source of Variation	df
Temperature	2
Recipe	1
Temperature$\times$Recipe	2
Between run error	6
Total	11

Multi-Stratum Designs

The second design produces information in two different *strata*. It gives the *split-unit* (or *split-plot*) analysis of variance.

Source of Variation	df
Temperature	2
Between run error	9
Run total	11
Recipe	1
Temperature$\times$Recipe	2
Between half-run error	9
Total	23

Multi-Stratum Designs

Multiple levels of units within the same experiment can arise in a number of ways, e.g.

- 6 varieties and 4 levels of fertiliser. It is convenient to apply fertilisers to entire strips of land.
- In laboratory or industrial experiments some factors just require changing the inputs into the system and can be changed several times per day, others require taking the equipment apart and reassembling it.
- In a clinical trial to compare the individual and combined effects of a drug and a skin cream. The cream can be applied to one hand, the drug has to be applied to a whole person.

Multi-Stratum Designs

There are three ways of thinking about split-unit designs:

- Think of the main units as the units. The units are split into subunits for some of the factors.
- Think of the subunits as the units. The randomization is restricted so that combinations with the same levels of some of the factors must appear together in main units.
- Think of the subunits as the units and the main units as blocks. Some of the factors' main effects are then confounded with block differences.

Either of the last two are consistent with the approach taken in this course and are essentially identical.

The confounding analogy implies that we should generally only use split-unit designs when we have to.

Notes

- The split-unit idea can be extended to split-split-units or even more levels of variation. There can also be crossed unit structures with factors applied to rows and/or columns.
- The simple least squares form of analysis used above is only possible because the designs in terms of the subunits are orthogonal with respect to the main units.

The general form of analysis is, as usual, determined by the randomization.

Multi-Stratum Designs

Having decided on our initial design, we randomly permute run labels to runs and randomly permute half-run labels to half-runs within runs.

The model under randomization is:

$$Y_{ij(kl)} = \mu + t_k + r_l + (tr)_{kl} + \rho_i + \epsilon_{ij},$$

where $V(\rho_i) = \sigma_1^2$ and $V(\epsilon_{ij}) = \sigma^2$.

As usual, I recommend starting with a separate analysis in each stratum, but it has become standard to assume normality and use REML-GLS.

Multi-Stratum Designs

Return to our example studying the effects of temperature and pressure on the yield of a reaction. It might be that it takes time for the temperature to stabilise, so that it is convenient to change temperature only once per day, leading to a split-unit design.

If we can have 4 runs per day the design might be

Day I		Day II		Day III		Day IV	
X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2
-1	-1	-1	-1	+1	-1	+1	-1
-1	-1	-1	0	+1	0	+1	0
-1	0	-1	0	+1	0	+1	+1
-1	+1	-1	+1	+1	+1	+1	+1

Multi-Stratum Designs

The randomization implies that there are two random effects, one for main plots with mean zero and variance σ_b^2, one for sub-plots with mean zero and variance σ^2.

A typical analysis might be to fit all estimable treatment effects up to second order and estimate the random effects by REML.

If we do this, we might find that the REML estimate of σ_b^2 is zero.

The main reason for this is that there is so little information on σ_b^2. Indeed under the randomization analysis it cannot be estimated in some unbalanced designs.

However, many packages will produce estimated standard errors for the main plot effects based on assuming $\sigma_b^2 = 0$.

Care is needed!

Designing unbalanced multi-stratum experiments

This is an area of much current research and some controversy. There are currently three different approaches:

- Choose a design from a class (or a number of classes) which ensure that GLS and OLS are equivalent. Ensures a simple and partly orthogonal analysis, but can be *very* inefficient.
- Find an optimal design, e.g. using one of the standard criteria, with respect to the GLS analysis for some prior estimates of the variance components. An efficient combined analysis will be obtained if the variance components are known, but it is unclear how good the designs will be in general. Maybe prior distributions for the variance components should be used.

Designing unbalanced multi-stratum experiments

- Separately find optimal designs in each stratum for the factors applied in that stratum, then block them optimally with respect to the higher strata, then adjust the design to ensure that interactions are estimated as well as possible. This ensures efficient analyses separately in each stratum, but is suboptimal for the combined analysis for most fixed values of the variance components.

I prefer the last of these, as it fits in with the general recommendation of initially performing separate analyses in each stratum.

A fully Bayesian approach to the design and analysis might be profitable. It will require somewhat informative priors for the variance components, but noninformative priors for the treatment effects might be acceptable.