The Theory of Designed Experiments

3. Restricted Randomization

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Restricted randomization

Note that, under randomization, inferences still depend on σ^2, the variance of unit effects around their mean. So if the units available are highly variable, estimates will be imprecise.

Therefore we often restrict the randomization, e.g. by blocking. Consider the randomized complete block design, i.e. we define *blocks* of units which contain each treatment once and randomly permute units to unit labels within the same block.

Restricted randomization

The model under randomization, for unit j in block i receiving treatment r, is

$$Y_{ij} = \mu + t_r + \sum_k \delta_{i;jk} d_{ik},$$

where the mean of d_{ik} in block i is not 0, but b_i, so we can write

$$Y_{ij} = \mu + t_r + b_i + \sum_k \delta_{i;jk} e_{ik},$$

where e_{ik} have mean 0 and variance σ^2 for each i.

This is again a linear model, which can be written

$$Y_{ij} = \mu + t_r + b_i + \epsilon_{ij}.$$

Restricted randomization

Note that σ^2 is now the variance of unit effects around the mean in the same block.

Hence the randomization should be restricted by blocking if we can define groups of units within which we expect unit effects to vary less than they do across the entire experiment.

The definition of blocks is determined by what we know about the units and not by the number of treatments we have. Therefore it is often necessary to use incomplete block designs, i.e. the number of units in a block is usually less than the number of treatments, except when there are only a few treatments.

Restricted randomization

In an incomplete block design, if possible we will randomly permute blocks to block labels as well as randomly permuting units to unit labels within each block. This ensures that each unit label has an equal chance of being applied to each unit.

This gives the model

$$
\begin{aligned}
Y_{ij} &= \mu + t_r + \sum_l \gamma_{il} \left(b_l + \sum_k \delta_{l;jk} e_{lk} \right) \\
&= \mu + t_r + \sum_l \gamma_{il} b_l + \sum_l \sum_k \gamma_{il} \delta_{l;jk} e_{lk},
\end{aligned}
$$

where b_l have mean 0 and variance σ_b^2 and e_{lk} have mean 0 and variance σ^2 for each l.

Restricted randomization

This can be written as

$$Y_{ij} = \mu + t_r + \beta_i + \epsilon_{ij},$$

where β_i is a random effect with variance σ_b^2 and ϵ_{ij} is a random effect with variance σ^2. This is a *linear mixed model*.

Note that each level of randomization (between blocks and within blocks) leads to a *random effect*.

This is often denoted, using the Wilkinson-Rogers notation, as `Blocks/Units`.

Restricted randomization

This extends to any number of nested levels of randomization (strata), e.g. if blocks are arranged in superblocks, we randomly permute superblocks, randomly permute blocks within superblocks and randomly permute units within blocks, giving

$$Y_{ijk} = \mu + t_r + \varsigma_i + \beta_{ij} + \epsilon_{ijk}.$$

This is expressed as `Superblocks/Blocks/Units`.

Restricted randomization

Randomization theory also works with two-way blocking (e.g. Latin squares) and other crossed unit structures.

In a row-column design, with one unit in each row×column combination, we would randomly permute rows and randomly permute columns.

Then, for the unit in row i and column j,

$$\begin{aligned} Y_{ij} &= \mu + t_r + \sum_k \sum_l \delta_{ik}\gamma_{jl}a_{kl} \\ &= \mu + t_r + \sum_k \delta_{ik} \sum_l \gamma_{jl}a_{kl}, \end{aligned}$$

where the mean of a_{kl} in row k is r_k.

Restricted randomization

So

$$\begin{aligned} Y_{ij} &= \mu + t_r + \sum_k \delta_{ik}(r_k + \sum_l \gamma_{jl} d_{kl}) \\ &= \mu + t_r + \sum_k \delta_{ik} r_k + \sum_l \gamma_{jl} \sum_k \delta_{ik} d_{kl}, \end{aligned}$$

where the mean of d_{kl} in column l is c_l. Hence,

$$\begin{aligned} Y_{ij} &= \mu + t_r + \sum_k \delta_{ik} r_k + \sum_l \gamma_{jl}(c_l + \sum_k \delta_{ik} e_{kl}) \\ &= \mu + t_r + \sum_k \delta_{ik} r_k + \sum_l \gamma_{jl} c_l + \sum_k \sum_l \delta_{ik}\gamma_{jl} e_{kl}. \end{aligned}$$

Restricted randomization

This is written as

$$Y_{ij} = \mu + t_r + \rho_i + \kappa_j + \epsilon_{ij},$$

where $V(\rho_i) = \sigma_r^2$, $V(\kappa_j) = \sigma_c^2$ and $V(\epsilon_{ij}) = \sigma^2$.

In Wilkinson-Rogers notation we write `Rows*Columns`.

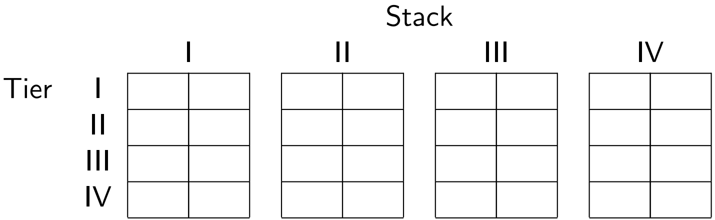
The nesting and crossing operators can be used to describe much more general unit structures defined by whatever randomization is done.

This can then be used to write down an appropriate linear mixed model, where

- there is a random effect for each stage of randomization;
- there is an additional random effect for the interaction of each pair of crossed effects.

Example

In a hen house, 32 chicken cages are arranged in 4 stacks of 4×2 cages. An experiment is randomized by permuting stacks, permuting tiers and permuting cages within each stack$\times$tier combination.



Unit structure is `(Stack*Tier)/Cage`.

Model is

$$Y_{ijk} = \mu + t_r + \varsigma_i + \tau_j + \upsilon_{ij} + \epsilon_{ijk}.$$

Information from Separate Strata

In incomplete block designs, the usual analysis, with fixed block effects, only uses information from within blocks. However, blocks have been randomized, so block totals contain information about the differences between treatments.

	Blocks					
	T_1 T_2	T_1 T_3	T_2 T_3	T_1 T_2	T_1 T_3	T_2 T_3
Total yield	10	12	1	11	12	2

Is T_1 clearly superior or did the randomization just lead to this allocation of blocks to block labels?

This is the *inter-block analysis*. It is separate from the usual intra-block analysis.

Notes

- The specific randomization determines the covariance structure in the derived linear mixed model.
- Randomization theory applies to *simple orthogonal block structures*, i.e. those made up from crossing and nesting of block factors each of which defines equally sized blocks.
- Randomization theory gives a separate analysis in each stratum, sometimes with information on treatments in different strata.
- Least squares analysis in each stratum gives minimum variance linear unbiased estimators of all treatment contrasts (using the information in that stratum) and minimum variance unbiased estimators of stratum variances.

Notes

- Randomization theory does *not* give a way of combining the separate analyses from each stratum.
- Nowadays, strata are usually combined by using REML-GLS analysis, but this depends on the normality of the random effects.
- In some cases, an analysis using fixed block effects might give a better approximation than an analysis using Normal random block effects.
- I recommend always doing the separate analyses in each stratum before any attempt is made to combine the analyses.

Other Forms of Analysis

The obvious alternative is to assume that the units are a sample from a (possibly structured) population of potential units.

This allows inferences to be made about the population of potential units, not just those used in the experiment, but is reliant on the additional assumptions about the nature of the population.

Then, for example, the randomized block model arises if we assume the units in different blocks are random samples from different populations. If the block populations are assumed to be a sample from a superpopulation, we have random block effects, as in the randomization analysis.

Other Forms of Analysis

If we randomize the experiment in a way that corresponds to the population structure, the two analyses are identical.

This is obviously a good idea!

Then only the generalization from the units used to the population depends on the assumptions made about the population. The analysis will be valid in any case.

What if we randomize in a different way to the population structure?

Other Forms of Analysis

If the population has the structure of a randomized complete block design, but we use a completely randomized design:

- if we analyse according to randomization, variances will be higher than necessary;
- if we analyse according to population, the design might be inefficient and the analysis is not justified by the randomization.

If the population has the structure of a completely randomized design, but we use a randomized complete block design:

- if we analyse according to the randomization, we lose only a few residual degrees of freedom;
- if we analyse according to the population, the analysis is not justified by randomization.

Other Forms of Analysis

We should randomize the experiment using all blocking factors that we think might be important in the population, as well as any others that are convenient.

We need only ensure that we have sufficient (10-20?) residual degrees of freedom.

In the analysis we should include all blocking factors defined by the randomization.

We might consider including in the analysis any other structure that we suspect the population might have (analysis of covariance).

Other Forms of Analysis

This demonstrates the intimate connection between models for the population, randomization and data analysis.

We should not ignore reasonable population models when designing the experiment and we should not ignore the design when modelling the data.

In particular, we should always include in the analysis all structure implied by the randomization. This allows us to see the need for any additional structure which will depend on stronger assumptions.

Other Forms of Analysis

Pure randomization-based analysis is frequentist, but it provides a natural baseline for a model-based analysis, whether likelihood or Bayesian.

The Bayesian viewpoint does not *object* to randomization and should not object to including blocking factors defined by the randomization in the model. Doing so helps to identify and justify the appropriate exchangeability assumptions.

I recommend always starting with the simple stratum-by-stratum randomization-based analysis. This allows us to separate:

- conclusions obtained from the randomization-based analysis (which are robust to assumptions);
- conclusions which depend on the model assumptions;
- conclusions which depend on a particular prior distribution.

Models for Unit Effects

Spatial analysis of field experiments and modelling time trends in laboratory experiments have become fashionable.

The reason is an alleged increase in precision. These methods involve replacing the unit effect e_i in our basic model with something more complex, e.g. an AR(1)$\times$AR(1) process in field experiments.

Some authors recommend excluding block effects from the model when doing spatial analysis.

Of course, I disagree! Even if the blocks have no physical meaning, the act of randomization means they should be included in the model, so that we can separate conclusions that depend on the modelling assumptions from those that do not.

Models for Unit Effects

The model implied by the randomization will often be adequate, so that no other strong assumptions are needed.

When the stronger assumptions lead to great increases in efficiency, it is often a sign that the experiment has been badly designed, e.g. with blocks that are too large.

Rescuing badly designed experiments seems to be the main purpose of these methods. However, the "rescue" is so dependent on assumptions that it is questionable whether these should still be called experiments.